Information Theory & Applications Workshop (ITA), UCSD, La Jolla, Feb. 2011

Analog-to-Digital Conversion Using Unstable Filters

Hans-Andrea Loeliger, Lukas Bolliger, Georg Wilckens, and Jonas Biveroni
Dept. of Information Technology & Electrical Engineering
ETH Zurich, CH-8092 Zurich, Switzerland

***Abstract*—A new approach to analog-to-digital conversion is proposed which combines unstable analog filters with digital Kalman filtering. The proposed approach subsumes sigma-delta converters, on which it offers a new perspective.**

***Index Terms*—Analog-to-digital conversion, Kalman filter, factor graphs, Gaussian message passing.**

I. Introduction

Analog-to-digital converters (ADCs) are key components in all digital electronic devices that process continuous-time signals [1]. In this paper, we propose a new approach to analog-to-digital conversion which subsumes and generalizes sigma-delta ADCs [1], [2].

The proposed ADCs will be called *unstable-filter ADCs*. Like most ADCs, the proposed ADCs have both an analog part and a digital part. The analog part contains an unstable continuous-time linear filter that is stabilized by discrete-time digital control signals. This combination may be viewed as taking the main idea of beta-expansion ADCs [3], [4] to continuous time.

The digital part of an unstable-filter ADC is essentially Kalman filtering based on a state-space model of the analog part, which we will describe in terms of factor graphs and Gaussian message passing as in [5]. We will also use results from [6], where the processing of continuous-time signals using factor graphs is described. For some missing details and proofs in [6], we refer to the fuller account in [7].

The paper is structured as follows. The proposed ADCs are presented in Sections II and III: Section II describes the analog part and Section III describes the digital part. Section IV addresses spectral shaping. Section V considers a special case which includes sigma-delta ADCs. The connection to beta-expansion ADCs is discussed in the appendix. Section VI concludes the paper.

II. The Analog Part

The analog part of an unstable-filter ADC has the structure shown in Fig. 1. The continuous-time signal $u(t)$ is fed into an unstable continuous-time linear system. The system is controlled (stabilized) by M control bits, which are obtained from M one-bit ADCs that monitor the system state at discrete points in time (as provided by a digital clock signal). These control bits will be denoted by $s_{1,k}, \ldots, s_{M,k}$, where the second index, k, is the discrete time. The linear system is also monitored by L additional flash ADCs with discrete-time output signals $y_{1,k}, \ldots, y_{L,k}$ which will be used to digitally track the state of the system.

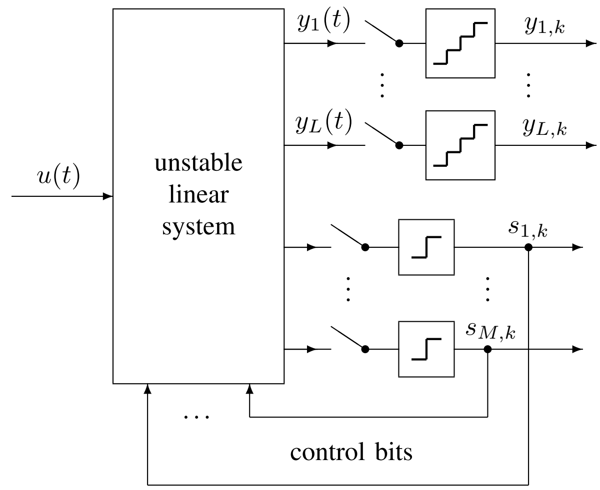

Fig. 1. Analog part (with digital control) of an unstable-filter ADC.

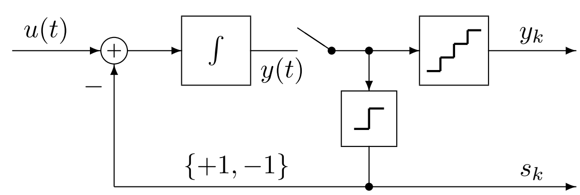

Fig. 2. A toy example.

A toy example of such a structure is shown in Fig. 2. In this example, the unstable linear system is a single integrator; there is only one control bit s_k and only one additional flash ADC that provides the discrete-time digital signal y_k. (The one-bit ADC that provides s_k may be omitted since s_k can be extracted from y_k.) If the continuous-time signal $u(t)$ is bounded by

$$|u(t)| \leq a < 1, \tag{1}$$

then the integrator output $y(t)$ is bounded by

$$|y(t)| < (1 + a)T \tag{2}$$

where T is the integration time between discrete-time samples. Note that this toy example looks like the analog part of a simple sigma-delta ADC.

We note that the unstable system provides an amplification or "expansion" similar to a sequential ADC. This perspective is elaborated in the appendix.

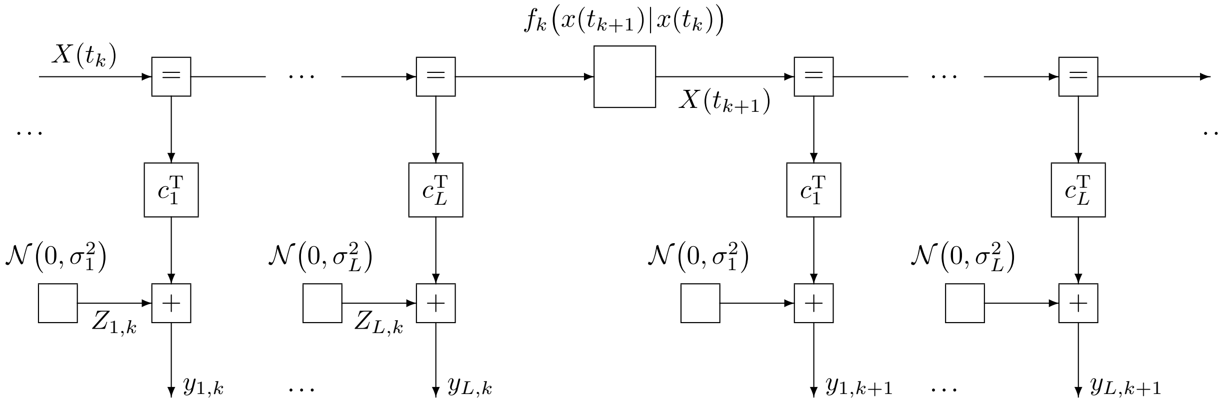

Fig. 3. Factor graph (as in [5], [6]) of the analog system in Fig. 1.

We will use a state space representation of such a system as follows. Let t_k, $k \in \mathbb{Z}$, be the discrete points in time when the quantized outputs $y_{\ell,k}$ and the control bits $s_{m,k}$ are sampled. Let $x(t) \in \mathbb{R}^N$ be the state of the system at time $t \in \mathbb{R}$ and let $\dot{x}(t)$ be its derivative with respect to t. Then

$$\dot{x}(t) = A_k x(t) + bu(t) + d_k \tag{3}$$

for $t_k \leq t < t_{k+1}$, where A_k is a real $N \times N$ matrix and where b and d_k are real column vectors. The matrix A_k and the vector d_k may depend on the control bits $s_{1,k}, \ldots, s_{M,k}$, but otherwise they are constant. The control bits influence the system only via their effect on A_k and d_k.

In addition, we have the continuous-time scalar outputs

$$y_\ell(t) = c_\ell^{\mathrm{T}} x(t) \tag{4}$$

with row vectors c_ℓ^{T}, $\ell = 1, \ldots L$. The digital outputs $y_{\ell,k}$ are quantized versions of $y_\ell(t_k)$.

In the toy example of Fig. 2, we have $N = L = M = 1$, $A_k = b = c_1 = 1$, and

$$d_k = \begin{cases} -1 & \text{if } s_k \text{ indicates ``}x(t_k) \geq 0\text{''} \\ +1 & \text{if } s_k \text{ indicates ``}x(t_k) < 0\text{''}, \end{cases} \tag{5}$$

where s_k is the single control bit.

III. The Digital Part

The digital part of an unstable-filter ADC estimates the input signal $u(t)$ from the digital signals $y_{\ell,k}$ and the control bits $s_{m,k}$. We propose to do this by modeling the input signal $u(t)$ as white Gaussian noise and forming the linear (or rather affine) minimum mean-squared-error (LMMSE) estimate of $u(t)$ based on the time-varying system model (3) and (4) and the observations $y_{\ell,k}$. This LMMSE estimate of $u(t)$ may be obtained by modeling the quantization error

$$z_{\ell,k} \triangleq y_{\ell,k} - y_\ell(t_k) \tag{6}$$

as white Gaussian noise (with the correct variance) and, based on this assumption, forming the MAP estimate of $u(t)$, which in turn may be obtained by means of Gaussian message passing in the factor graph of the system model (3)–(4), cf. [5, Section V] and [6].

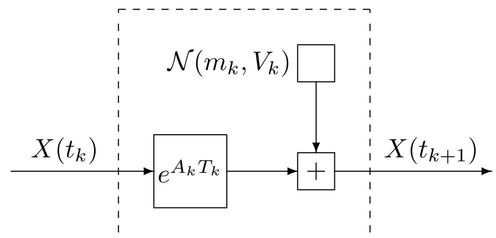

Fig. 4. A factor graph of $f_k\big(x(t_{k+1})|x(t_k)\big)$.

The corresponding Forney factor graph (in the style of [5]) is shown in Fig. 3. Since the quantities $x(t)$, $z_{\ell,k}$, etc., are now random variables, we will now denote them by capital letters, i.e., $X(t)$, $Z_{\ell,k}$, etc.; the observations $y_{\ell,k}$ are still denoted by lowercase letters. The centerpiece of Fig. 3 is the factor $f_k\big(x(t_{k+1})|x(t_k)\big)$, which is the conditional density of $X(t_{k+1})$ for any fixed $X(t_k) = x(t_k)$ according to (3), assuming that the analog input signal $U(t)$ is white Gaussian noise with variance σ_U^2. The other parts of Fig. 3 represent (4), (6), and the assumption that, for any fixed ℓ, the quantization error $Z_{\ell,k}$ is white Gaussian noise.

The function $f_k\big(x(t_{k+1})|x(t_k)\big)$ can itself be represented by the factor graph shown in Fig. 4 with $T_k \triangleq t_{k+1} - t_k$,

$$m_k = \int_0^{T_k} e^{A_k t} d_k \, dt, \tag{7}$$

and

$$V_k = \sigma_U^2 \int_0^{T_k} e^{A_k t} b b^{\mathrm{T}} (e^{A_k t})^{\mathrm{T}} \, dt. \tag{8}$$

Since Fig. 3 is a linear Gaussian factor graph, the MAP estimate of $U(t)$ (for arbitrary discrete times t) may be obtained by Gaussian message passing in this factor graph as in [5] and [6]. In particular, from eq. (I.5) of [6], the desired estimate of $u(t)$, for any $t \in \mathbb{R}$, is

$$\hat{u}(t) = \sigma_U^2 b^{\mathrm{T}} \left(\overrightarrow{V}_{X(t)} + \overleftarrow{V}_{X(t)} \right)^{-1} \left(\overleftarrow{m}_{X(t)} - \overrightarrow{m}_{X(t)} \right) \tag{9}$$

where $\overrightarrow{m}_{X(t)}$ and $\overrightarrow{V}_{X(t)}$ are the parameters (the mean vector and the covariance matrix, respectively) of the forward mes-

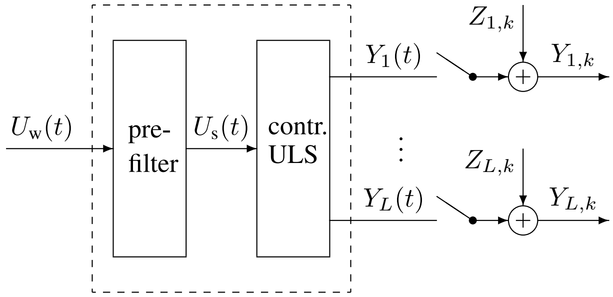

Fig. 5. Extended system model with a prefilter before the unstable linear system (ULS).

sage and where $\overleftarrow{m}_{X(t)}$ and $\overleftarrow{V}_{X(t)}$ are the parameters of the backwards message along the edge $X(t)$.

IV. Spectral Shaping

The LMMSE estimate (9) is marked by an implicit spectral shaping by the continuous-time system model as pointed out in [6].

It may sometimes be desirable, however, to control the spectrum of the estimate more explicitly. This can be achieved by augmenting the system model of Section III with a separate stable prefilter as shown in Fig. 5. This prefilter represents an assumption on the spectrum of the physical input signal $U_\mathrm{s}(t)$; it is not physically present in the analog part of the ADC. The hypothetical input signal $U_\mathrm{w}(t)$ in Fig. 5 is white Gaussian noise. In this case, we produce as digital ADC output the LMMSE estimate of the filtered signal $U_\mathrm{s}(t)$ rather than an estimate of $U_\mathrm{w}(t)$.

V. Sigma-Delta Converters as a Special Case

In an important special case, the matrix $A_k = A$ in (3) is constant and the vector d_k in (3) can be written as $d_k = Ds_k$ for some constant matrix D and for $s_k \triangleq (s_{1,k}, \ldots, s_{M,k})^\mathrm{T} \in \{+1, -1\}^M$. Eq. (3) thus becomes

$$\dot{x}(t) = Ax(t) + bu(t) + Ds_k. \tag{10}$$

E.g., the toy example of Fig. 2 can be represented in this way with $A = 1$ and $D = -1$.

We then have a time-invariant system model, and the LMMSE estimate $\hat{u}(t)$ (or $\hat{u}_\mathrm{s}(t)$ as in Section IV) depends *linearly* both on the observations $y_{\ell,k}$ and on the control bits $s_{m,k}$. The digital processing of Section III thus becomes a time-invariant linear filter, which is the usual digital processing in sigma-delta converters. In this way, unstable-filter ADCs subsume sigma-delta ADCs that rely on one-bit flash ADCs.

If we generalize the control bits $s_{m,k}$ in Fig. 1 to multi-level symbols, all sigma-delta ADCs may be viewed as special cases of unstable-filter ADCs.

VI. Conclusion

We have proposed a new approach to analog-to-digital conversion. Like most ADCs, the proposed ADCs have both an analog part and a digital part. The analog part provides digital *evidence* about the continuous-time signal $u(t)$ that is to be converted; the digital part controls the analog part and *infers* $u(t)$ from the evidence provided by the analog part.

The analog part of the proposed ADCs contains an unstable linear filter that is controlled digitally. This combination may be viewed as taking the main idea of sequential ADCs—expansion and control—to continuous time (cf. the appendix).

The proposed approach subsumes sigma-delta ADCs, on which it offers a new perspective with a clear conceptual separation between control (by means of the control bits $s_{m,k}$) and system tracking (based on the quantized observations $y_{\ell,k}$).

In addition to these general features, the proposed approach offers the following advantages and opportunities:

- Digital estimates of the continuous-time signal $u(t)$ may be obtained for arbitrary discrete points in time, independently of the sampling times t_k [6].
- The thresholds for the control bits $s_{m,k}$ (Fig. 1) need not be accurate.
- Noise in the analog circuitry (Fig. 1) can be handled mathematically by extending the input signal $u(t)$ to a vector of white-Gaussian-noise signals whose additional components model the noise.
- Nonlinearities in the analog filter can be handled by extended Kalman filtering, i.e., by iterative processing with a linearized model based on a tentative estimate of the state trajectory $x(t)$.

Appendix

Expansion and Control in Ideal Sequential ADCs

The ideal sequential ADC converts a real number x, $0 \leq x < 1$, which is given in the analog domain, into a sequence of bits $s_1, s_2, \ldots, s_N \in \{0, 1\}$ by means of the following recursion: beginning with $x_1 \triangleq x$, we compute

$$x_{k+1} = 2x_k - s_k \tag{11}$$

where

$$s_k = \begin{cases} 0, & \text{if } x_k < 1/2 \\ 1, & \text{if } x_k \geq 1/2, \end{cases} \tag{12}$$

and we obtain

$$\hat{x} = \sum_{k=1}^{N} s_k 2^{-k} + 2^{-(N+1)} \tag{13}$$

as the digital estimate of x.

Note that both (11) and the comparison in (12) need to be computed with high precision in the analog domain, which is very costly. In consequence, this seemingly elegant procedure has little, if any, practical value as an ADC.

The beta-expansion ADC is a generalization of the ideal sequential ADC where (11) is replaced by

$$x_{k+1} = \beta x_k - s_k \tag{14}$$

where β is a real number such that $1 < \beta \leq 2$ and where

$$s_k = \begin{cases} 0, & \text{if } \beta x_k < 1 \\ 0 \text{ or } 1, & \text{if } 1 \leq \beta x_k < \frac{1}{\beta - 1} \\ 1, & \text{if } \beta x_k \geq \frac{1}{\beta - 1} \end{cases} \tag{15}$$

The corresponding digital estimate of x is

$$\hat{x} = \sum_{k=1}^{N} s_k \beta^{-k} + \beta^{-(N+1)}. \tag{16}$$

In the middle case of (15), s_k may be chosen freely, which means that no high-precision comparator is required to determine s_k. However, the computation (14) still needs to be carried out with high precision in the analog domain. (Beta-expansion ADCs with low-precision analog computation are discussed in [8].)

These ideal sequential ADCs use a sequence of "expansions" (multiplication by 2 or by β, respectively) to magnify x and a sequence of "control actions" (the subtraction of s_k) to confine x_k within some practical range. An unstable-filter ADC also uses these two elements (expansion and control), but it implements the expansion in continuous time. As with the beta-expansion ADC, the control bits of an unstable-filter ADC need not be computed with high precision.

REFERENCES

[1] D. A. Johns and K. Martin, *Analog Integrated Circuit Design,* New York: Wiley, 1997.

[2] P. M. Aziz, H. V. Sorensen, and J. V. Spiegel, "An overview of sigma-delta converters," *IEEE Signal Proc. Mag.*, vol. 13, no. 1, pp. 61–84, Jan. 1996.

[3] I. Daubechies, R. A. DeVore, C. S. Güntürk, and V. A. Vaishampayan, "Beta expansions: a new approach to digitally corrected A/D conversion," *Proc. 2002 IEEE Int. Symp. Circuits and Systems (ISCAS),* May 26–29, 2002, pp. II-784–II-787.

[4] I. Daubechies and Özgür Yılmaz, "Robust and practical analog-to-digital conversion with exponential precision," *IEEE Trans. Inform. Theory,* vol. 52, no. 8, Aug. 2006.

[5] H.-A. Loeliger, J. Dauwels, Junli Hu, S. Korl, Li Ping, and F. R. Kschischang, "The factor graph approach to model-based signal processing," *Proceedings of the IEEE,* vol. 95, no. 6, pp. 1295–1322, June 2007.

[6] L. Bolliger, H.-A. Loeliger, and C. Vogel, "Simulation, MMSE estimation, and interpolation of sampled continuous-time signals using factor graphs," 2010 Information Theory & Applications Workshop, UCSD, La Jolla, CA, USA, Jan. 31 – Feb. 5, 2010.

[7] L. Bolliger, H.-A. Loeliger, and C. Vogel, "MMSE estimation and interpolation of sampled continuous-time signals using factor graphs," submitted.

[8] J. Biveroni and H.-A. Loeliger, "On sequential analog-to-digital conversion with low-precision components," 2008 Information Theory & Applications Workshop, UCSD, La Jolla, CA, Jan. 27 – Feb. 1, 2008.